FAKULTETA ZA NARAVOSLOVJE IN TEHNOLOGIJO
Oddelek za fiziko

Peter Prelovšek in Ivan Kuščer

ZBIRKA FORMUL ZA VEKTORJE IN TENZORJE

Društvo matematikov, fizikov in astronomov SRS
Ljubljana 1982

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UDK 51

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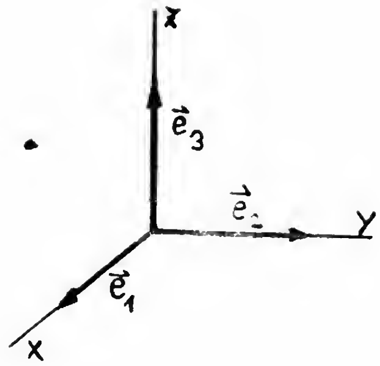
1. Kartezične komponente

vektor:

$\vec{a} = (a_1, a_2, a_3)$, $\quad a_1 = a_x = \vec{a}\cdot\vec{e}_1$ itd.

$\vec{e}_1 = (1,0,0)$ itd.



tenzor:

$$\underline{A} = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix} = \sum_{i,j} A_{ij}\, \vec{e}_i \otimes \vec{e}_j$$

$$A_{ij} = \vec{e}_i\cdot(\underline{A}\vec{e}_j) = (\vec{e}_i\underline{A})\cdot\vec{e}_j$$

adjungirani (transponirani) tenzor:

$$\underline{A}^{\dagger} = \begin{pmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{pmatrix}, \quad A^{\dagger}_{ij} = A_{ji}, \qquad (\underline{A}^{\dagger})^{\dagger} = \underline{A}$$

2. Seštevanje

$$(\vec{a} + \vec{b})_i = a_i + b_i, \qquad (\underline{A} + \underline{B})_{ij} = A_{ij} + B_{ij}$$

3. Produkti

skalarni:

$$\vec{a}\cdot\vec{b} = a_1b_1 + a_2b_2 + a_3b_3 = ab\cos(\vec{a},\vec{b}) = \vec{b}\cdot\vec{a}$$

vektorski:

$$\vec{a}\times\vec{b} = \begin{vmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \left(\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}, \begin{vmatrix} a_3 & a_1 \\ b_3 & b_1 \end{vmatrix}, \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \right) = -\vec{b}\times\vec{a}$$

$$|\vec{a}\times\vec{b}| = ab\sin(\vec{a},\vec{b})$$

tenzorski:

$$\vec{a}\otimes\vec{b} = \begin{pmatrix} a_1b_1 & a_1b_2 & a_1b_3 \\ a_2b_1 & a_2b_2 & a_2b_3 \\ a_3b_1 & a_3b_2 & a_3b_3 \end{pmatrix} = (\vec{b}\otimes\vec{a})^{\dagger}$$

produkt tenzorjev:

$$\underline{\underline{A}}\,\underline{\underline{B}} = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix} \begin{pmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{pmatrix} = \begin{pmatrix} \sum_j A_{1j}B_{j1}, \dots \\ \dots \\ \dots \end{pmatrix}$$

$$(\underline{\underline{A}}\,\underline{\underline{B}})_{ik} = \sum_j A_{ij}B_{jk}\ ,$$

$$(\underline{\underline{A}}\,\underline{\underline{B}})^{\dagger} = \underline{\underline{B}}^{\dagger}\underline{\underline{A}}^{\dagger}$$

produkt tenzorja z vektorjem:

$$\underline{A}\vec{b} = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \Big(\sum_j A_{1j}b_j, \dots, \dots\Big) = \vec{b}\underline{A}^{\dagger}$$

$$\vec{b}\underline{A} = (b_1, b_2, b_3) \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix} = \Big(\sum_i b_i A_{i1}, \dots, \dots\Big) = \underline{A}^{\dagger}\vec{b}$$

$$(\underline{A}\vec{b})_i = \sum_j A_{ij}b_j, \qquad (\vec{b}\underline{A})_j = \sum_i b_i A_{ij}$$

vektorski produkt tenzorja z vektorjem:

$$\underline{A}\times\vec{b} = \begin{pmatrix} A_{12}b_3 - A_{13}b_2, & A_{13}b_1 - A_{11}b_3, & A_{11}b_2 - A_{12}b_1 \\ A_{22}b_3 - A_{23}b_2, & A_{23}b_1 - A_{21}b_3, & A_{21}b_2 - A_{22}b_1 \\ A_{32}b_3 - A_{33}b_2, & A_{33}b_1 - A_{31}b_3, & A_{31}b_2 - A_{32}b_1 \end{pmatrix} = \underline{A}(\vec{b}\times\underline{1})$$

$$\vec{b}\times\underline{A} = -\,(\underline{A}^{\dagger}\times\vec{b})^{\dagger}$$

vsi produkti so distributivni

4. Posebni tenzorji

enotni (identiteta): $\underline{1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \sum_i \vec{e}_i \otimes \vec{e}_i$

izotropen: $\underline{A} = \lambda\underline{1}$

simetričen (sebi adjungiran): $\underline{A} = \underline{A}^{\dagger}$

antisimetričen in ustrezni vektor:

$$\underline{A} = -\underline{A}^{\dagger} = \vec{a}\times\underline{1} = \begin{pmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{pmatrix}, \qquad \vec{a} = (-A_{23}, A_{13}, -A_{12})$$

$$\vec{a}\times\vec{b} = \underline{A}\vec{b} = (\vec{a}\times\underline{1})\vec{b}$$

unitaren: $\underline{A}\underline{A}^{\dagger} = \underline{A}^{\dagger}\underline{A} = \underline{1}$

5. Vrtenje koordinatnega sistema

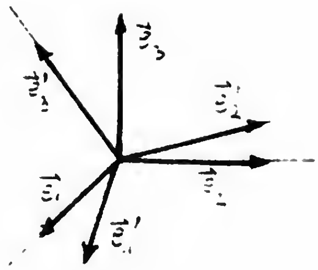
$$\vec{a} = \sum_j a_j \vec{e}_j = \sum_j (\vec{a}\cdot\vec{e}_j)\vec{e}_j$$

$$a_i' = \vec{a}\cdot\vec{e}_i' = \sum_j (\vec{e}_i'\cdot\vec{e}_j)(\vec{a}\cdot\vec{e}_j) = \sum_j \alpha_{ij}\, a_j$$

$$\alpha_{ij} = \vec{e}_i'\cdot\vec{e}_j = \cos(\hat{e}_i', \vec{e}_j)$$

$$\underline{\alpha}\underline{\alpha}^{\dagger} = \underline{\alpha}^{\dagger}\underline{\alpha} = \underline{1}$$

$$A_{ij}' = \sum_{k,l} \alpha_{ik}\, A_{kl}\, \alpha_{lj}^{\dagger}$$



6. Invariante pri vrtenju

$$a \equiv |\vec{a}| = \sqrt{\vec{a}\cdot\vec{a}} = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

$$Tr(\underline{A}) = A_{11} + A_{22} + A_{33}$$

$$\det(\underline{A}) = \begin{vmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{vmatrix}$$

$$\det(\underline{A}\underline{B}) = \det(\underline{A})\det(\underline{B})$$

$$\det(\vec{a}\otimes\vec{b}) = 0$$

$$Tr(\vec{a}\otimes\vec{b}) = \vec{a}\cdot\vec{b}$$

7. Lastne vrednosti sebi adjungiranega tenzorja

$\underline{A}\vec{\omega} = \lambda\vec{\omega} \Rightarrow \exists\vec{\omega} \neq 0$ če $\det(\underline{A} - \lambda\underline{1}) = 0$

če $\underline{A} = \underline{A}^{\dagger} \Longrightarrow$ 3 realni koreni $\lambda_I, \lambda_{II}, \lambda_{III}$ (lastne vrednosti $\underline{A}$-ja)

ustrezni lastni vektorji: $\vec{\omega}_I, \vec{\omega}_{II}, \vec{\omega}_{III}$

jih normiramo in, če treba, ortogonaliziramo: $\vec{\omega}_i\cdot\vec{\omega}_j = \delta_{ij}$

$$Tr(\underline{A}) = \lambda_I + \lambda_{II} + \lambda_{III}$$

$$\det(\underline{A}) = \lambda_I\, \lambda_{II}\, \lambda_{III}$$

8. Večkratni produkti

$$\vec a\cdot(\vec b\times\vec c) = (\vec a\times\vec b)\cdot\vec c = \begin{vmatrix} a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$\vec a\times(\vec b\times\vec c) = (\vec a\cdot\vec c)\vec b - (\vec a\cdot\vec b)\vec c$$

$$(\vec a\times\vec b)\times\vec c = (\vec a\cdot\vec c)\vec b - (\vec b\cdot\vec c)\vec a$$

$$\vec a\times(\vec b\times\vec c) + \vec b\times(\vec c\times\vec a) + \vec c\times(\vec a\times\vec b) = 0$$

$$\vec a = \vec u(\vec a\cdot\vec u) + \vec u\times(\vec a\times\vec u), \text{ če } |\vec u| = 1$$

$$(\vec a\times\vec b)\cdot(\vec c\times\vec d) = \begin{vmatrix} \vec a\cdot\vec c & \vec a\cdot\vec d\\ \vec b\cdot\vec c & \vec b\cdot\vec d \end{vmatrix}$$

$$(\vec a\otimes\vec b)\vec c = \vec a(\vec b\cdot\vec c) = \vec c(\vec b\otimes\vec a)$$

$$(\vec a\otimes\vec b)\times\vec c = \vec a\otimes(\vec b\times\vec c)$$

produkt tenzorjev je asociativen, produkt tenzorjev in vektorjev pa ne vedno:

$$(\underline{A}\,\underline{B})\underline{C} = \underline{A}(\underline{B}\,\underline{C}) = \underline{A}\,\underline{B}\,\underline{C}$$

$$\vec a\cdot(\underline{B}\vec c) = (\vec a\underline{B})\cdot\vec c,\quad \underline{A}(\underline{B}\vec c) = (\underline{A}\,\underline{B})\vec c = \underline{A}\,\underline{B}\vec c,\quad \vec a(\underline{B}\,\underline{C}) = (\vec a\underline{B})\underline{C} = \vec a\underline{B}\,\underline{C}$$

toda na splošno $\underline{A}(\vec b\underline{C}) \neq (\underline{A}\vec b)\underline{C}$ in seveda $(\vec a\cdot\vec b)\underline{C} \neq \vec a\cdot(\vec b\underline{C})$

9. Definicije odvodov

gradient skalarja:

$$df(\vec r) = d\vec r\cdot\nabla f = d\vec r\cdot\text{grad } f,\qquad \vec n\cdot\nabla f = \left(\frac{\partial f}{\partial s}\right)_{\|\vec n},\ \text{če } |\vec n| = 1$$

odvod vektorja v predpisani smeri:

$$d\vec v(\vec r) = (d\vec r\cdot\nabla)\vec v,\qquad (\vec n\cdot\nabla)\vec v = \left(\frac{\partial\vec v}{\partial s}\right)_{\|\vec n},\ \text{če } |\vec n| = 1$$

divergenca vektorja:

$$\nabla\cdot\vec v(\vec r) = \text{div } \vec v(\vec r) = \lim_{\max|\vec r\,'-\vec r|\to 0}\frac{1}{V}\oint d\vec S\,'\cdot\vec v(\vec r\,')$$

divergenca tenzorja:

$$\nabla\,\underline{A}(\vec r) = \text{div } \underline{A}(\vec r) = \lim_{\max|\vec r\,'-\vec r|\to 0}\frac{1}{V}\oint d\vec S\,'\underline{A}(\vec r\,')$$

(integrala po površini , ki objema prostornino V, vektor $d\vec S\,'$ obrnjen pravokotno navzven)

rotor:

$$\vec{n}\cdot(\nabla\times\vec{v}(\vec{r})) = \vec{n}\cdot\mathrm{rot}\ \vec{v}(\vec{r}) = \lim_{\max|\vec{r}'-\vec{r}|\to 0} \frac{1}{S}\oint \overrightarrow{dr'}\cdot\vec{v}(\vec{r}')$$

(integral po ravni zanki, ki objema ploščino S in ima normalo $\vec{n}$, obrnjeno po svedrskem pravilu)

gradient vektorja:

$$\vec{n}(\nabla\otimes\vec{v}) = (\vec{n}\cdot\nabla)\vec{v}$$

simetriziran gradient vektorja (deformacija):

$$\mathrm{def}\ \vec{v} = \frac{1}{2}\left[\nabla\otimes\vec{v} + (\nabla\otimes\vec{v})^{\dagger}\right]$$

$$\mathrm{Tr}(\mathrm{def}\ \vec{v}) = \nabla\cdot\vec{v}$$

10. Odvodi posebnih izrazov in produktov

$$\nabla f(g(\vec{r})) = f'(g)\,\nabla g(\vec{r}), \qquad \nabla f(r) = f'(r)\,\vec{r}/r$$

$$\nabla(\vec{a}\cdot\vec{r}) = \vec{a}$$

$$\nabla\cdot\vec{r} = 3, \qquad \nabla\times\vec{r} = 0, \qquad (\vec{c}\cdot\nabla)\vec{r} = \vec{c}$$

$$\nabla\cdot(\vec{a}\times\vec{r}) = 0, \qquad \nabla\times(\vec{a}\times\vec{r}) = 2\vec{a}, \qquad (\vec{c}\cdot\nabla)(\vec{a}\times\vec{r}) = \vec{a}\times\vec{c}$$

$$\nabla(f\,\underline{1}) = \nabla f, \qquad \nabla(\vec{v}\times\underline{1}) = \nabla\times\vec{v}$$

$$\nabla(fg) = f\,\nabla g + g\,\nabla f$$

$$\nabla(\vec{u}\cdot\vec{v}) = (\vec{u}\cdot\nabla)\vec{v} + (\vec{v}\cdot\nabla)\vec{u} + \vec{u}\times(\nabla\times\vec{v}) + \vec{v}\times(\nabla\times\vec{u})$$

$$\nabla(v^2/2) = (\vec{v}\cdot\nabla)\vec{v} + \vec{v}\times(\nabla\times\vec{v})$$

$$\nabla\cdot(f\vec{v}) = f\nabla\cdot\vec{v} + \vec{v}\cdot\nabla f$$

$$\nabla\cdot(\vec{u}\times\vec{v}) = \vec{v}\cdot(\nabla\times\vec{u}) - \vec{u}\cdot(\nabla\times\vec{v})$$

$$\nabla\times(f\vec{v}) = f\,\nabla\times\vec{v} - \vec{v}\times\nabla f$$

$$\nabla\times(\vec{u}\times\vec{v}) = (\vec{v}\cdot\nabla)\vec{u} - (\vec{u}\cdot\nabla)\vec{v} + \vec{u}(\nabla\cdot\vec{v}) - \vec{v}(\nabla\cdot\vec{u})$$

$$(\vec{u}\cdot\nabla)f\vec{v} = \vec{v}(\vec{u}\cdot\nabla f) + f\,(\vec{u}\cdot\nabla)\vec{v}$$

$$\nabla(\vec{u}\otimes\vec{v}) = \vec{v}(\nabla\cdot\vec{u}) + (\vec{u}\cdot\nabla)\vec{v}$$

$$\nabla(f\,\underline{A}) = (\nabla f)\underline{A} + f\,(\nabla\underline{A})$$

$$\nabla\cdot(\underline{A}\vec{v}) = \vec{v}\cdot(\nabla\underline{A}) + \mathrm{Tr}(\underline{A}\ \mathrm{def}\ \vec{v}),\ \text{če}\ \underline{A} = \underline{A}^\dagger$$

11. Drugi odvodi

$$\nabla^2 f = \nabla\cdot\nabla f$$

$$\nabla^2\vec{v} = \nabla(\nabla\otimes\vec{v}) = \nabla(\nabla\cdot\vec{v}) - \nabla\times(\nabla\times\vec{v})$$

$$\nabla\times(\nabla f) = 0$$

$$\nabla\cdot(\nabla\times\vec{v}) = 0$$

$$\nabla(\mathrm{def}\ \vec{v}) = \frac{1}{2}\nabla^2\vec{v} + \frac{1}{2}\nabla(\nabla\cdot\vec{v})$$

12. Integralne formule

Gaussovi izreki (integral po zaključeni ploskvi = = integral po objeti prostornini):

$$\oint d\vec{S}\ f = \int dV\ \nabla f$$

$$\oint d\vec{S}\cdot\vec{v} = \int dV\ \nabla\cdot\vec{v}$$

z $\vec{v} = f\,\nabla g - g\,\nabla f$ dobimo Greenovo formulo:

$$\oint d\vec{S}\cdot(f\,\nabla g - g\,\nabla f) = \int dV\ (f\,\nabla^2 g - g\,\nabla^2 f).$$

$$\oint d\vec{S}\times\vec{v} = \int dV\ \nabla\times\vec{v}$$

$$\oint (\vec{a}\cdot d\vec{S})\ \vec{v} = \int dV\ (\vec{a}\cdot\nabla)\vec{v},\quad \text{če}\ \vec{a} = \text{const}$$

$$\oint d\vec{S}\ \underline{A} = \int dV\ \nabla\underline{A}\ .$$

Stokesovi izreki (integral po zanki = integral po objeti ploskvi):

$$\oint d\vec{r}\cdot\vec{v} = \int d\vec{S}\cdot(\nabla\times\vec{v})$$

$$\oint d\vec{r}\times\vec{v} = \int (d\vec{S}\times\nabla)\times\vec{v}$$

$$\oint d\vec{r}\ f = \int d\vec{S}\times\nabla f$$

13. Odvodi v raznih pravokotnih koordinatnih sistemih

splošne koordinate: $q_1(\vec{r})$, $q_2(\vec{r})$, $q_3(\vec{r})$

$$\nabla q_i = \vec{e}_i(\vec{r})/h_i(\vec{r}),\quad \vec{e}_i\cdot\vec{e}_j = \delta_{ij},\quad \vec{v}(\vec{r}) = \sum_i v_i(\vec{r})\,\vec{e}_i(\vec{r})$$

$$d\vec{r} = \sum_i h_i\vec{e}_i dq_i,\quad dV = h_1h_2h_3dq_1dq_2dq_3$$

$$\nabla f = \frac{1}{h_1}\frac{\partial f}{\partial q_1}\vec{e}_1 + \frac{1}{h_2}\frac{\partial f}{\partial q_2}\vec{e}_2 + \frac{1}{h_3}\frac{\partial f}{\partial q_3}\vec{e}_3$$

$$\nabla\cdot\vec{v} = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial q_1}(h_2h_3v_1) + \frac{\partial}{\partial q_2}(h_3h_1v_2) + \frac{\partial}{\partial q_3}(h_1h_2v_3)\right]$$

$$\nabla\times\vec{v} = \frac{1}{h_2h_3}\left[\frac{\partial}{\partial q_2}(h_3v_3) - \frac{\partial}{\partial q_3}(h_2v_2)\right]\vec{e}_1 + \frac{1}{h_3h_1}\left[\frac{\partial}{\partial q_3}(h_1v_1) - \right.$$

$$\left. - \frac{\partial}{\partial q_1}(h_3v_3)\right]\vec{e}_2 + \frac{1}{h_1h_2}\left[\frac{\partial}{\partial q_1}(h_2v_2) - \frac{\partial}{\partial q_2}(h_1v_1)\right]\vec{e}_3$$

$$\nabla^2 f = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial q_1}\left(\frac{h_2h_3}{h_1}\frac{\partial f}{\partial q_1}\right) + \frac{\partial}{\partial q_2}\left(\frac{h_3h_1}{h_2}\frac{\partial f}{\partial q_2}\right) + \frac{\partial}{\partial q_3}\left(\frac{h_1h_2}{h_3}\frac{\partial f}{\partial q_3}\right)\right]$$

kartezične koordinate: $q_1 = x$, $q_2 = y$, $q_3 = z$

$$\nabla f = \frac{\partial f}{\partial x}\vec{e}_x + \frac{\partial f}{\partial y}\vec{e}_y + \frac{\partial f}{\partial z}\vec{e}_z$$

$$\nabla\cdot\vec{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

$$\nabla\times\vec{v} = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\vec{e}_x + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\vec{e}_y + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\vec{e}_z$$

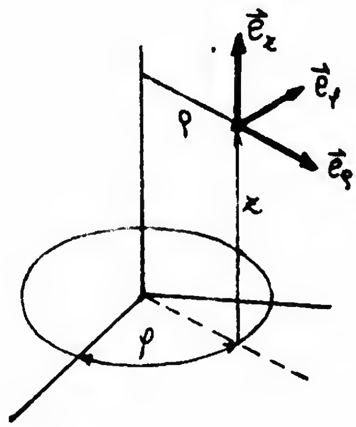
$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

$$\nabla^2\vec{v} = \frac{\partial^2\vec{v}}{\partial x^2} + \frac{\partial^2\vec{v}}{\partial y^2} + \frac{\partial^2\vec{v}}{\partial z^2}$$

$$\nabla\underline{A} = \left(\frac{\partial A_{11}}{\partial x} + \frac{\partial A_{21}}{\partial y} + \frac{\partial A_{31}}{\partial z},\ \ldots,\ \ldots\right)$$

$$(\nabla\otimes\vec{v})_{ij} = \frac{\partial v_j}{\partial x_i},\qquad \mathrm{def}\,\vec{v} = \begin{pmatrix} \frac{\partial v_x}{\partial x} & \frac{1}{2}\left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}\right) & \frac{1}{2}\left(\frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x}\right) \\ \frac{1}{2}\left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}\right) & \frac{\partial v_y}{\partial y} & \frac{1}{2}\left(\frac{\partial v_y}{\partial z} + \frac{\partial v_z}{\partial y}\right) \\ \frac{1}{2}\left(\frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x}\right) & \frac{1}{2}\left(\frac{\partial v_y}{\partial z} + \frac{\partial v_z}{\partial y}\right) & \frac{\partial v_z}{\partial z} \end{pmatrix}$$

cilindrične koordinate:



$q_1 = \rho,\ q_2 = \varphi,\ q_3 = z$

$h_1 = 1,\ h_2 = \rho\ ,\ h_3 = 1$

$dV = \rho\, d\rho\, d\varphi\, dz$

$$\nabla f = \frac{\partial f}{\partial \rho}\vec{e}_\rho + \frac{1}{\rho}\frac{\partial f}{\partial \varphi}\vec{e}_\varphi + \frac{\partial f}{\partial z}\vec{e}_z$$

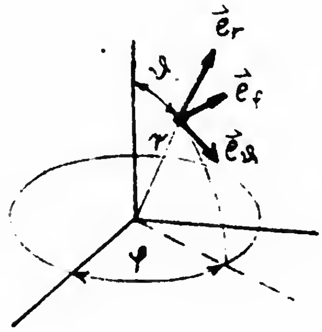
$$\nabla\cdot\vec{v} = \frac{1}{\rho}\frac{\partial(\rho v_\rho)}{\partial \rho} + \frac{1}{\rho}\frac{\partial v_\varphi}{\partial \varphi} + \frac{\partial v_z}{\partial z}$$

$$\nabla\times\vec{v} = \left(\frac{1}{\rho}\frac{\partial v_z}{\partial \varphi} - \frac{\partial v_\varphi}{\partial z}\right)\vec{e}_\rho + \left(\frac{\partial v_\rho}{\partial z} - \frac{\partial v_z}{\partial \rho}\right)\vec{e}_\varphi + \frac{1}{\rho}\left(\frac{\partial(\rho v_\varphi)}{\partial \rho} - \frac{\partial v_\rho}{\partial \varphi}\right)\vec{e}_z$$

$$\nabla^2 f = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial f}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 f}{\partial \varphi^2} + \frac{\partial^2 f}{\partial z^2}$$

$$\nabla^2\vec{v} = \left(\nabla^2 v_\rho - \frac{v_\rho}{\rho^2} - \frac{2}{\rho^2}\frac{\partial v_\varphi}{\partial \varphi}\right)\vec{e}_\rho + \left(\nabla^2 v_\varphi - \frac{v_\varphi}{\rho^2} + \frac{2}{\rho^2}\frac{\partial v_\rho}{\partial \varphi}\right)\vec{e}_\varphi + \nabla^2 v_z\ \vec{e}_z$$

krogelne koordinate:



$q_1 = r,\ q_2 = \vartheta,\ q_3 = \varphi$

$h_1 = 1,\ h_2 = r\ ,\ h_3 = r\sin\vartheta$

$dV = r^2 dr\, d(\cos\vartheta)\, d\varphi$

$$\nabla f = \frac{\partial f}{\partial r}\vec{e}_r + \frac{1}{r}\frac{\partial f}{\partial \vartheta}\vec{e}_\vartheta + \frac{1}{r\sin\vartheta}\frac{\partial f}{\partial \varphi}\vec{e}_\varphi$$

$$\nabla\cdot\vec{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\vartheta}\frac{\partial}{\partial \vartheta}(\sin\vartheta\, v_\vartheta) + \frac{1}{r\sin\vartheta}\frac{\partial v_\varphi}{\partial \varphi}$$

$$\nabla\times\vec{v} = \frac{1}{r\sin\vartheta}\left[\frac{\partial}{\partial \vartheta}(\sin\vartheta\, v_\varphi) - \frac{\partial v_\vartheta}{\partial \varphi}\right]\vec{e}_r + \\ + \frac{1}{r}\left[\frac{1}{\sin\vartheta}\frac{\partial v_r}{\partial \varphi} - \frac{\partial(r v_\varphi)}{\partial r}\right]\vec{e}_\vartheta + \frac{1}{r}\left[\frac{\partial(r v_\vartheta)}{\partial r} - \frac{\partial v_r}{\partial \vartheta}\right]\vec{e}_\varphi$$

$$\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\vartheta}\frac{\partial}{\partial \vartheta}\left(\sin\vartheta\frac{\partial f}{\partial \vartheta}\right) + \frac{1}{r^2\sin^2\vartheta}\frac{\partial^2 f}{\partial \varphi^2}$$

PREDGOVOR

Zbirka podaja v strnjeni obliki tiste formule za vektorje in za tenzorje 2. reda (kvadratne matrike), ki se pogosto rabijo. Namenjena je uporabniku, ki pojme že pozna, tako da mu zadoščajo prav skopa besedna pojasnila. Pisava brez komponent, ki ni odvisna od izbire koordinatnega sistema, naj vzpodbuja k nazorni predstavi.

Znamenja se večji del ravnajo po prevladujoči rabi v fizikalni literaturi. Skalarni, vektorski in tenzorski produkt dveh vektorjev zaznamujeva s $\cdot$, $\times$ in $\otimes$. Produkte s skalarji pa piševa brez ločila in ravno tako matrične produkte (tenzorja s tenzorjem in tenzorja z vektorjem). Izogibava se razlikovanja "vektorjev vrstic" in "vektorjev stolpičev".

Omejila sva se na 3-dimenzionalni evklidski prostor; vendar se bo uporabnik sam znašel z nekaterimi posplošitvami. V več dimenzijah završemo vektorski produkt, medtem ko ohranimo vse ostalo. V kompleksnih prostorih je treba npr. pri drugem faktorju skalarnega produkta in pri komponentah adjungiranega tenzorja (ki se ne ujema več s transponiranim tenzorjem) vstavljati konjugirano kompleksne vrednosti, takole:

$$\vec{u}\cdot\vec{v} = \sum_i u_i v_i^* = (\vec{v}\cdot\vec{u})^* , \qquad A^\dagger_{ij} = A^*_{ji} .$$

V Hilbertovem prostoru, kjer tenzorjem ustrezajo operatorji, pišemo po navadi $(\vec{u},\vec{v})$ ali $\langle\vec{v}|\vec{u}\rangle$ namesto $\vec{u}\cdot\vec{v}$.

Izmed uporabljenih virov naj omeniva naslednje:

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Rada bi se zahvalila Egonu Zakrajšku za temeljit pregled rokopisa in za nasvete, s katerimi nama je pomagal odpraviti več nedoslednosti.

Peter Prelovšek, Ivan Kuščer

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Izdajata: Oddelek za fiziko FNT in
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Odgovorni urednik Janez Strnad

13.a.

Peter Prelovšek in Ivan Kuščer

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Strokovni pregled Egon Zakrajšek

Izdalo Društvo matematikov, fizikov in astronomov SRS
Urednik Ciril Velkovrh

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